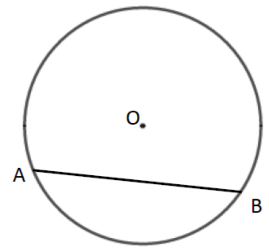
Circle theorems

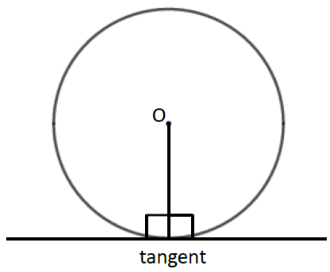
A LEVEL LINKS
Scheme of work: 2b. Circles – equation of a circle, geometric problems on a grid

Key points

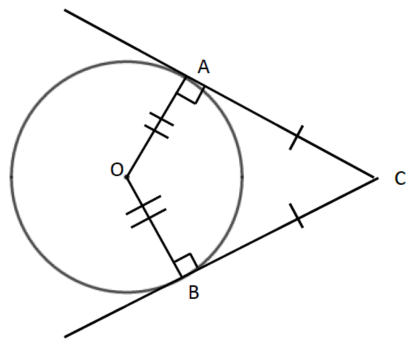
- A chord is a straight line joining two points on the circumference of a circle.
  So AB is a chord.



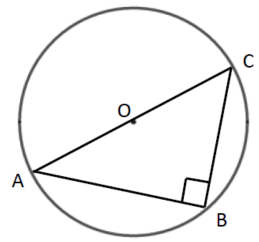
- A tangent is a straight line that touches the circumference of a circle at only one point.
  The angle between a tangent and the radius is 90°.



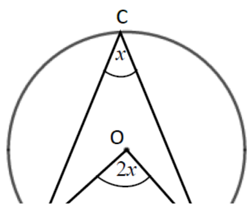
- Two tangents on a circle that meet at a point outside the circle are equal in length.
  So AC = BC.



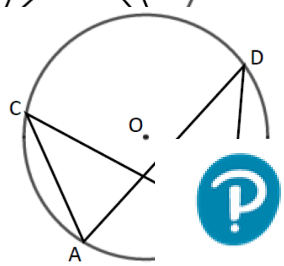
- The angle in a semicircle is a right angle.
  So angle ABC = 90°.



- When two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference.
  So angle AOB = 2 × angle ACB.

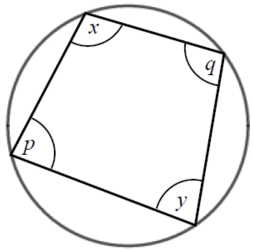


- Angles subtended by the same arc at the circumference are equal. This means that angles

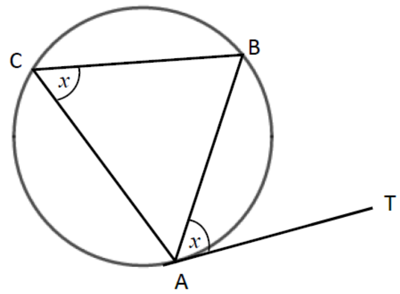


in the same segment are equal.
So angle ACB = angle ADB and
angle CAD = angle CBD.

- A cyclic quadrilateral is a quadrilateral with all four vertices on the circumference of a circle. Opposite angles in a cyclic quadrilateral total 180°. So $x + y = 180°$ and $p + q = 180°$.

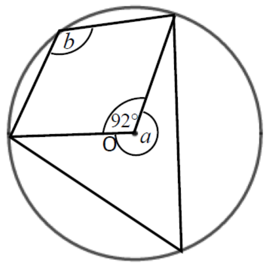


- The angle between a tangent and chord is equal to the angle in the alternate segment, this is known as the alternate segment theorem. So angle BAT = angle ACB.



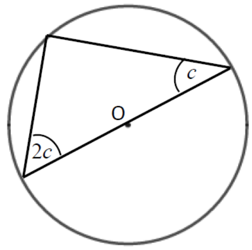
Examples

Example 1 Work out the size of each angle marked with a letter. Give reasons for your answers.



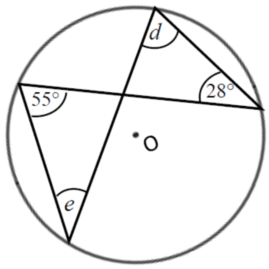
Angle $a = 360° - 92°$ $= 268°$ as the angles in a full turn total 360°. Angle $b = 268° \div 2$ $= 134°$ as when two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference.	**1** The angles in a full turn total 360°. **2** Angles a and b are subtended by the same arc, so angle b is half of angle a.

Example 2 Work out the size of the angles in the triangle.
Give reasons for your answers.



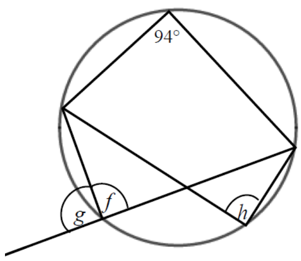
Angles are 90°, $2c$ and c. $90° + 2c + c = 180°$ $90° + 3c = 180°$ $3c = 90°$ $c = 30°$ $2c = 60°$ The angles are 30°, 60° and 90° as the angle in a semi-circle is a right angle and the angles in a triangle total 180°.	**1** The angle in a semicircle is a right angle. **2** Angles in a triangle total 180°. **3** Simplify and solve the equation.

Example 3 Work out the size of each angle marked with a letter.
Give reasons for your answers.



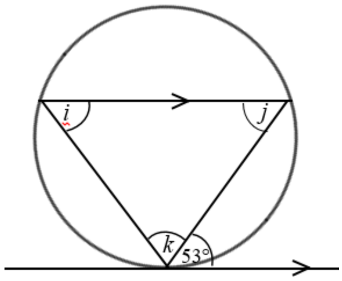
Angle d = 55° as angles subtended by the same arc are equal. Angle e = 28° as angles subtended by the same arc are equal.	**1** Angles subtended by the same arc are equal so angle 55° and angle d are equal. **2** Angles subtended by the same arc are equal so angle 28° and angle e are equal.

Example 4 Work out the size of each angle marked with a letter.
Give reasons for your answers.



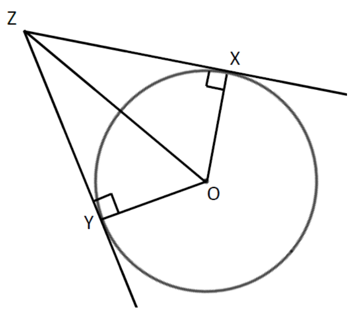
Angle $f = 180° - 94°$ $= 86°$ as opposite angles in a cyclic quadrilateral total 180°.	**1** Opposite angles in a cyclic quadrilateral total 180° so angle 94° and angle f total 180°. *(continued on next page)*
Angle $g = 180° - 86°$ $= 84°$ as angles on a straight line total 180°. Angle h = angle f = 86° as angles subtended by the same arc are equal.	**2** Angles on a straight line total 180° so angle f and angle g total 180°. **3** Angles subtended by the same arc are equal so angle f and angle h are equal.

Example 5 Work out the size of each angle marked with a letter. Give reasons for your answers.



Angle i = 53° because of the alternate segment theorem. Angle j = 53° because it is the alternate angle to 53°. Angle $k = 180° - 53° - 53°$ $= 74°$ as angles in a triangle total 180°.	**1** The angle between a tangent and chord is equal to the angle in the alternate segment. **2** As there are two parallel lines, angle 53° is equal to angle j because they are alternate angles. **3** The angles in a triangle total 180°, so $i + j + k = 180°$.

Example 6 XZ and YZ are two tangents to a circle with centre O. Prove that triangles XZO and YZO are congruent.

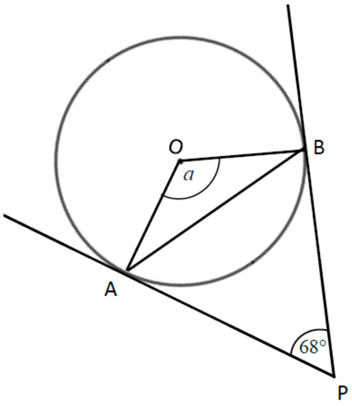


Angle OXZ = 90° and angle OYZ = 90° as the angles in a semicircle are right angles. OZ is a common line and is the hypotenuse in both triangles. OX = OY as they are radii of the same circle. So triangles XZO and YZO are congruent, RHS.	For two triangles to be congruent you need to show one of the following. ● All three corresponding sides are equal (SSS). ● Two corresponding sides and the included angle are equal (SAS). ● One side and two corresponding angles are equal (ASA). ● A right angle, hypotenuse and a shorter side are equal (RHS).

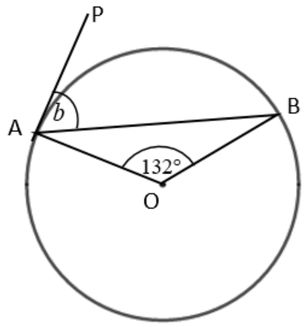
Practice

1 Work out the size of each angle marked with a letter.
Give reasons for your answers.

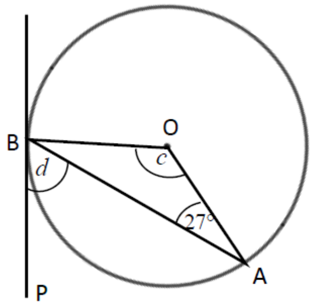
a



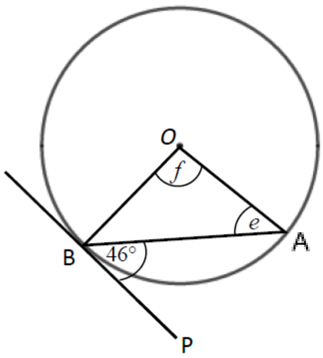
b



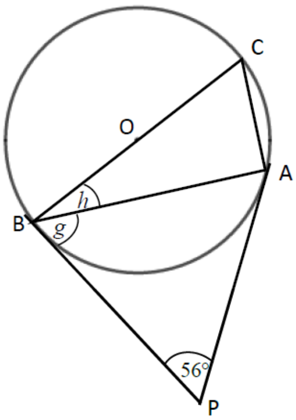
c



d

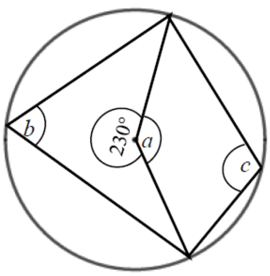


e



2 Work out the size of each angle marked with a letter.
Give reasons for your answers.

a



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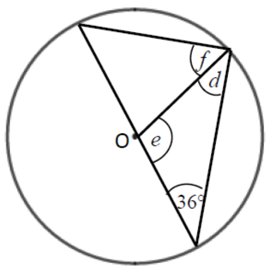

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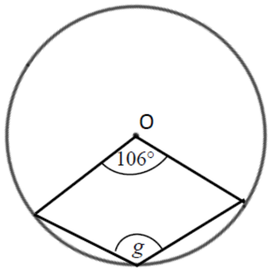
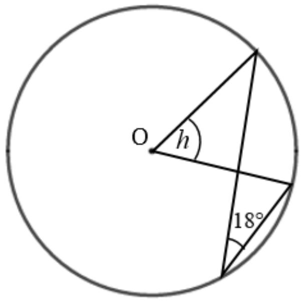


> **Hint**
>
> The reflex angle at point O and angle *g* are subtended by the same arc. So the reflex angle is twice the size of angle *g*.

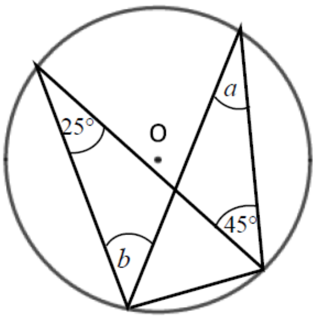
d



> **Hint**
>
> Angle 18° and angle *h* are subtended by the same arc.

3 Work out the size of each angle marked with a letter.
Give reasons for your answers.

a



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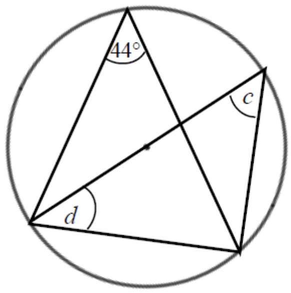
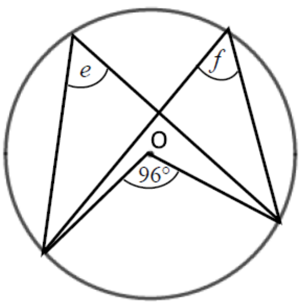


> **Hint**
>
> One of the angles is in a semicircle.

c



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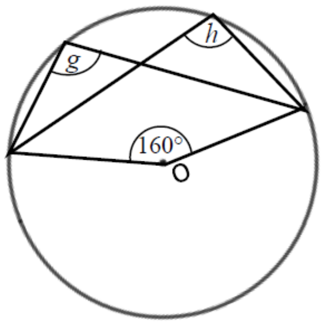
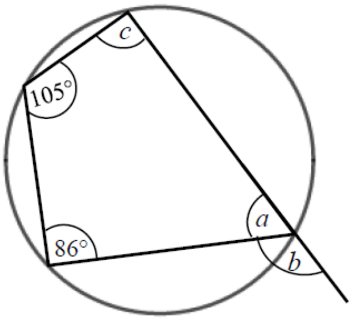


4 Work out the size of each angle marked with a letter.
Give reasons for your answers.

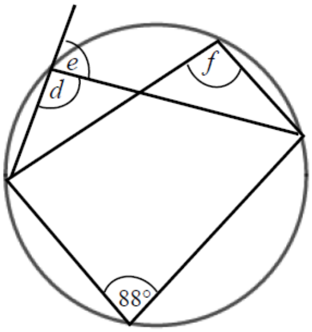
a



Hint

An exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

b



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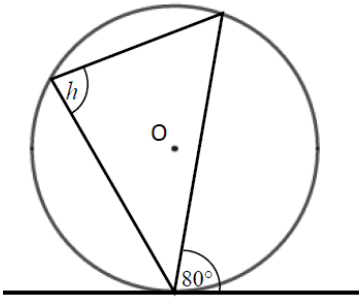
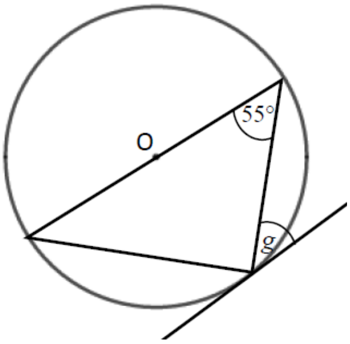


d



Hint

One of the angles is in a semicircle.

Extend

5 Prove the alternate segment theorem.

Answers

1 **a** $a = 112°$, angle OAP = angle OBP = 90° and angles in a quadrilateral total 360°.

b $b = 66°$, triangle OAB is isosceles, Angle OAP = 90° as AP is tangent to the circle.

c $c = 126°$, triangle OAB is isosceles.
$d = 63°$, Angle OBP = 90° as BP is tangent to the circle.

d $e = 44°$, the triangle is isosceles, so angles e and angle OBA are equal. The angle OBP = 90° as BP is tangent to the circle.
$f = 92°$, the triangle is isosceles.

e $g = 62°$, triangle ABP is isosceles as AP and BP are both tangents to the circle.
$h = 28°$, the angle OBP = 90°.

2 **a** $a = 130°$, angles in a full turn total 360°.
$b = 65°$, the angle at the centre of a circle is twice the angle at the circumference.
$c = 115°$, opposite angles in a cyclic quadrilateral total 180°.

b $d = 36°$, isosceles triangle.
$e = 108°$, angles in a triangle total 180°.
$f = 54°$, angle in a semicircle is 90°.

c $g = 127°$, angles at a full turn total 360°, the angle at the centre of a circle is twice the angle at the circumference.

d $h = 36°$, the angle at the centre of a circle is twice the angle at the circumference.

3 **a** $a = 25°$, angles in the same segment are equal.
$b = 45°$, angles in the same segment are equal.

b $c = 44°$, angles in the same segment are equal.
$d = 46°$, the angle in a semicircle is 90° and the angles in a triangle total 180°.

c $e = 48°$, the angle at the centre of a circle is twice the angle at the circumference.
$f = 48°$, angles in the same segment are equal.

d $g = 100°$, angles at a full turn total 360°, the angle at the centre of a circle is twice the angle at the circumference.
$h = 100°$, angles in the same segment are equal.

4 **a** $a = 75°$, opposite angles in a cyclic quadrilateral total 180°.
$b = 105°$, angles on a straight line total 180°.
$c = 94°$, opposite angles in a cyclic quadrilateral total 180°.

b $d = 92°$, opposite angles in a cyclic quadrilateral total 180°.
$e = 88°$, angles on a straight line total 180°.
$f = 92°$, angles in the same segment are equal.

c $h = 80°$, alternate segment theorem.

d $g = 35°$, alternate segment theorem and the angle in a semicircle is 90°.

5 Angle BAT = x.

Angle OAB = 90° − x because the angle between the tangent and the radius is 90°.

OA = OB because radii are equal.

Angle OAB = angle OBA because the base of isosceles triangles are equal.

Angle AOB = 180° − (90° − x) − (90° − x) = $2x$ because angles in a triangle total 180°.

Angle ACB = $2x$ ÷ 2 = x because the angle at the centre is twice the angle at the circumference.